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AFDELING ZUIVERE WISKUNDE (DEPARTMENT OF PURE MATHEMATICS) ZN 77/77 NOVEMBER

A.E. BROUWER

THE WORST COVERING OF POINTS BY PERMUTATIONS

amsterdam 1977

stichting
mathematisch
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Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O).

AMS(MOS) subject classification scheme (1970): 05B40

The worst covering of points by permutations

by

A.E. Brouwer

ABSTRACT

We show that for $n \geq 3$ the cardinality of a largest minimal cover of points by permutations is $n(n-2)$.

KEY WORDS & PHRASES: *permutation-design, covering.*

INTRODUCTION

M. Deza introduced the concepts of packing and covering of permutations, analogues of the corresponding concepts for sets.

A collection P of permutations of $I_n = \{1,2,\dots,n\}$ is called a t-packing (resp. t-cover) if for each injection $f\colon T \to I_n$ (where $|T| = t$ and $T \subset I_n$) there is at most (resp. at least) one $\pi \in P$ such that $\pi|T = f$.

A minimal cover is a cover such that none of its elements can be removed; a worst covering is a minimal cover with maximal cardinality. (And likewise we have the concepts of maximal packing and worst packing.) If we represent the permutation

$$\begin{pmatrix} 1 & 2 & \dots & n \\ \pi_1 & \pi_2 & \dots & \pi_n \end{pmatrix}$$

by the row $\pi_1\pi_2\dots\pi_n$, then we are looking for $N\times n$ matrices with N as large as possible such that each column contains all the numbers $1,2,\dots,n$ while each row contains an element that is unique in its column. Deza told me that $n(n-2)$ is an upper bound for N while $n(n-2)$ can be achieved for $n = 3,4,5$. We shall see that indeed $N = n(n-2)$ is possible for all $n \geq 3$.

Now we know everything about permutation 1-designs: The worst packing, best packing and best covering all have n elements (the corresponding matrix being a Latin square).

About permutation 2-designs we have less information; Deza showed that a perfect permutation 2-design (an optimal 2-packing that is at the same time an optimal 2-cover) is essentially the same object as a projective plane of order n.

For worst designs one can prove that for $n = 4$

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{matrix}$$

is (the unique) worst packing, and the $4! - 4 = 20$ remaining permutations form (the unique) worst covering. (Note that the worst 2-packing for $n = 4$ has less elements than the worst 2-packing for $n = 3$!)

The worst 1-cover

For $n \leq 3$ we have:

$$n = 2: \quad \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \qquad\qquad n = 3: \quad \begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}$$

(in these cases the best and the worst 1-cover coincide).

For $n > 3$ we have:

If a column contains n unique elements, then there are only n rows.

If a column contains n-1 unique elements then by induction there are at most $(n-1) + (n-1)(n-3) = (n-1)(n-2) < n(n-2)$ rows.

If each column contains at most n-2 unique elements then there are at most $n(n-2)$ rows.

Hence for $n \geq 3$: $N \leq n(n-2)$

and if equality holds (and $n > 3$) then each column contains exactly n-2 unique elements.

Example for $n = 5$ (the unique elements are underlined):

$$\begin{array}{ccccc l}
\underline{2} & 1 & 3 & 4 & 5 & \pi_{21} \\
1 & \underline{3} & 2 & 4 & 5 & \pi_{22} \\
1 & 2 & \underline{4} & 3 & 5 & .. \\
1 & 2 & 3 & \underline{5} & 4 & .. \\
5 & 2 & 3 & 4 & \underline{1} & \pi_{25} \\
\\
\underline{3} & 1 & 2 & 4 & 5 & \pi_{31} \\
1 & \underline{4} & 2 & 3 & 5 & .. \\
1 & 2 & \underline{5} & 3 & 4 & .. \\
5 & 2 & 3 & \underline{1} & 4 & .. \\
5 & 1 & 3 & 4 & \underline{2} & \pi_{35} \\
\\
\underline{4} & 1 & 2 & 3 & 5 & \pi_{41} \\
1 & \underline{5} & 2 & 3 & 4 & .. \\
5 & 2 & \underline{1} & 3 & 4 & .. \\
5 & 1 & 3 & \underline{2} & 4 & .. \\
5 & 1 & 2 & 4 & \underline{3} & \pi_{45}
\end{array}$$

Generally for $2 \le i \le n-1$ and $1 \le j \le n$ we define the permutation π_{ij} by

$$\pi_{ij}(k) = \begin{cases} k & \text{if } i \le k-j \le n-1 \\ k-1 & \text{if } 1 \le k-j \le i-1 \\ \underline{i+j-1} & \text{if } j = k \end{cases}$$

where all arithmetic is done mod n.

It is easily seen that

(i) each π_{ij} is a permutation of I_n, and

(ii) in column k all permutations have k or k-1 except the permutations π_{ik} which have

$$\pi_{ik}(k) = k+i-1 \qquad (i=2,\ldots,n-1)$$

so that each of them is necessary.

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